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Pattern avoidance and fiber bundle structures on Schubert varieties

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Abstract. We give a permutation pattern avoidance criteria for determining when the projection map from the flag variety to a Grassmannian induces a fiber bundle structure on a Schubert variety. In particular, we show that a Schubert variety has such a fiber bundle structure if and only if the corresponding permutation avoids the split patterns 3|12 and 23|1. We also show that a Schubert variety is an iterated fiber bundle of Grassmannian Schubert varieties if and only if the corresponding permutation avoids (non-split) patterns 3412, 52341, and 635241.

Résumé. Nous donnons un schéma de permutation des critères d'évitement pour déterminer quand la carte de projection du drapeau de la variété à un Grassmannienne induit une structure de faisceau de fibres sur une variété de Schubert. En particulier, nous montrons qu'une variété de Schubert a une telle structure de faisceau de fibres si et seulement si la permutation correspondante évite les motifs fendus 3|12 et 23|1. Nous montrons aussi qu'une variété de Schubert est un faisceau de fibres itéré de variétés Grassmannienne Schubert si et seulement si la permutation correspondante évite (non-fractionnées) modèles 3412, 52341 et 635241.

Keywords: Permutation pattern avoidance, Schubert varieties

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field and let

$$\mathrm{F}\ell(n) := \{V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{K}^n \mid \dim(V_i) = i\}$$

denote the complete flag variety on $\mathbb{K}^n$. For each $r \in \{1, \ldots, n-1\}$, let $\mathrm{Gr}(r,n)$ denote the Grassmannian of r-dimensional subspaces of $\mathbb{K}^n$ and consider the natural projection map

$$\pi_r : \mathrm{F}\ell(n) \twoheadrightarrow \mathrm{Gr}(r,n) \tag{1.1}$$

given by $\pi_r(V_\bullet) = V_r$. It is easy to see that the projection π_r is a fiber bundle on $\mathrm{F}\ell(n)$ with fibers isomorphic to $\mathrm{F}\ell(r) \times \mathrm{F}\ell(n-r)$. The goal of this paper is to give a pattern

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avoidance criteria for when the map π_r restricted to a Schubert variety of $\mathrm{F}\ell(n)$ is also a fiber bundle.

Fix a basis $\{e_1, \dots, e_n\}$ of $\mathbb{K}^n$ and let $E_i := \mathrm{span}\langle e_1, \dots, e_i \rangle$. Each permutation $w = w(1) \cdots w(n)$ of the symmetric group $\mathfrak{S}_n$ defines a Schubert variety

$$X_w := \{V_\bullet \in \mathrm{F}\ell(n) \mid \dim(E_i \cap V_j) \geq r_w[i,j]\}$$

where $r_w[i,j] := \#\{k \leq j \mid w(k) \leq i\}$. For details on the geometry of the map π_r restricted to X_w, see Lemma 2.4 and Proposition 2.6.

Theorem 1.1. *Let $r < n$ and $w \in \mathfrak{S}_n$. The projection π_r restricted to X_w is a Zariski-locally trivial fiber bundle if and only if w avoids the split patterns $3|12$ and $23|1$ with respect to position r.*

If a permutation avoids a split pattern with respect to every position $r < n$, then that permutation avoids the pattern in the classical sense. For a precise definition of split pattern avoidance, see Definition 2.2. Pattern avoidance has been used to combinatorially describe many geometric properties of Schubert varieties. Most notably, Lakshmibai and Sandhya prove that a Schubert variety X_w is smooth if and only if w avoids the patterns 3412 and 4231 [3]. Pattern avoidance has been used to characterize many other geometric properties on Schubert varieties as well. For a survey of these results see [1].

1.1 Complete parabolic bundle structures

For any positive integer n, define the set $[n] := \{1, \dots, n\}$. The varieties $\mathrm{F}\ell(n)$ and $\mathrm{Gr}(r,n)$ are the extreme examples in the collection of partial flag varieties on $\mathbb{K}^n$. For any subset $\mathbf{a} := \{a_1 < \cdots < a_k\} \subseteq [n-1]$, define the partial flag variety

$$\mathrm{F}\ell(\mathbf{a}, n) := \{V_\bullet^{\mathbf{a}} := V_{a_1} \subset V_{a_2} \subset \cdots \subset V_{a_k} \subseteq \mathbb{K}^n \mid \dim(V_{a_i}) = a_i\}.$$

If $\mathbf{b} \subseteq \mathbf{a}$, then there is a natural projection map $\pi_{\mathbf{b}}^{\mathbf{a}} : \mathrm{F}\ell(\mathbf{a}, n) \twoheadrightarrow \mathrm{F}\ell(\mathbf{b}, n)$ given by $\pi_{\mathbf{b}}^{\mathbf{a}}(V_\bullet^{\mathbf{a}}) = V_\bullet^{\mathbf{b}}$. Note that the map $\pi_r = \pi_{\{r\}}^{[n-1]}$ from (1.1). Any permutation $\sigma = \sigma(1) \cdots \sigma(n-1) \in \mathfrak{S}_{n-1}$ defines a collection of nested subsets

$$\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_{n-2} \subset \sigma_{n-1} = [n-1] \quad \text{where} \quad \sigma_i := \{\sigma(1), \dots, \sigma(i)\}.$$

The maps $\pi_{\sigma_{i-1}}^{\sigma_i}$ induce an iterated fiber bundle structure on the complete flag variety

$$\mathrm{F}\ell(n) \overset{\pi_{\sigma_{n-2}}^{[n-1]}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_{n-2}, n) \overset{\pi_{\sigma_{n-3}}^{\sigma_{n-2}}}{\twoheadrightarrow} \cdots \overset{\pi_{\sigma_2}^{\sigma_3}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_2, n) \overset{\pi_{\sigma_1}^{\sigma_2}}{\twoheadrightarrow} \mathrm{F}\ell(\sigma_1, n) \twoheadrightarrow pt \tag{1.2}$$

where the fibers of each map $\pi_{\sigma_{i-1}}^{\sigma_i}$ are isomorphic to Grassmannians.

Definition 1.2. *Let $w \in \mathfrak{S}_n$. We say X_w has a **complete parabolic bundle structure** if there is a permutation $\sigma \in \mathfrak{S}_{n-1}$ such that the maps $\pi^{\sigma_i}_{\sigma_{i-1}}$ induce an iterated fiber bundle structure on the Schubert variety*

$$X_w = X_{n-1} \overset{\pi^{[n-1]}_{\sigma_{n-2}}}{\twoheadrightarrow} X_{n-2} \overset{\pi^{\sigma_{n-2}}_{\sigma_{n-3}}}{\twoheadrightarrow} \cdots \overset{\pi^{\sigma_3}_{\sigma_2}}{\twoheadrightarrow} X_2 \overset{\pi^{\sigma_2}_{\sigma_1}}{\twoheadrightarrow} X_1 \twoheadrightarrow pt \tag{1.3}$$

where $X_i := \pi^{[n-1]}_{\sigma_i}(X_n) \subseteq \mathrm{F}\ell(\sigma_i, n)$. In other words, each map $\pi^{\sigma_i}_{\sigma_{i-1}} : X_i \twoheadrightarrow X_{i-1}$ is a Zariski-locally trivial fiber bundle.

Some Schubert varieties do not have complete parabolic bundle structures. The smallest such Schubert variety is X_{3412}. When $\mathbb{K} = \mathbb{C}$, Ryan showed that any smooth Schubert variety has complete parabolic bundle structure [6]. Wolper later generalized this result to include Schubert varieties over any algebraically closed field [7]. Combining these results with the Lakshmibai-Sandhya smoothness criteria, we have:

Theorem 1.3. ([6, 7, 3]) *If w avoids patterns* 3412 *and* 4231*, then X_w has a complete parabolic bundle structure.*

An analogous result to Theorem 1.3 holds true for rationally smooth Schubert varieties of any finite type [5]. We remark that the converse of Theorem 1.3 is false. For example, the permutation $\sigma = 213$ induces a complete parabolic bundle structure on X_{4231}. One application of Theorem 1.1 is a pattern avoidance characterization of Schubert varieties that have complete parabolic bundle structures.

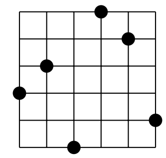
Theorem 1.4. *The permutation w avoids patterns* 3412, 52341 *and* 635241 *if and only if the Schubert variety X_w has a complete parabolic bundle structure.*

The key property used to prove both Theorems 1.1 and 1.4 is the notion of a Billey-Postnikov (BP) decomposition $w = vu$ of a permutation (see Proposition 2.6 for the definition). The term BP decomposition was originally used in [4] to describe a certain factorization condition on the Poincaré polynomials of w, v, u observed by Billey and Postnikov in [2]. Since then, several equivalent conditions have been given to describe this property (see [5, Section 4]).

2 Preliminaries

For any integers $m < n$, define the interval $[m, n] := \{m, m+1, \ldots, n\}$ and let $[n] := [1, n]$. We now denote the symmetric group $W := \mathfrak{S}_n$ and will denote permutations $w \in W$ using one-line notation $w = w(1)w(2)\cdots w(n)$. Diagrammatically, we draw a representation of the permutation matrix of w with nodes marking the points $(w(i), i)$ using the convention that $(1, 1)$ marks the upper left corner.

Example 2.1. *The permutation* $w = 436125$ *corresponds to the matrix:*



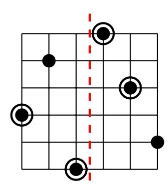
A **split pattern** $w = w_1|w_2 \in W$ is a divided permutation where $w_1 = w(1)\cdots w(j)$ and $w_2 = w(j+1)\cdots w(n)$ for some $j \in [n-1]$. We use split patterns to make the following modified definition of pattern containment and avoidance.

Definition 2.2. *Let* $k, r \leq n$. *Let* $w = w(1)\cdots w(n)$ *and* $u = u(1)\cdots u(j)|u(j+1)\cdots u(k)$. *We say* w ***contains the split pattern*** u *with respect to position* r *if there exists a sequence* $(i_1 < \cdots < i_k) \subseteq [n]$ *such that*

1. $w(i_1)\cdots w(i_k)$ *has the same relative order as* u
2. $i_j \leq r < i_{j+1}$.

If w *does not contain* u *with respect to position* r, *then we say* w ***avoids the split pattern*** u *with respect to position* r.

Example 2.3. *Let* $w = 426135$ *and* $u = 34|12$. *Then* w *contains the split pattern* u *with respect to position* $r = 3$, *but avoids the split pattern* u *with respect to all other positions.*



Note that part (1) of Definition 2.2 is the usual definition of pattern containment. It is easy to see that if w avoids a split pattern u with respect to all $r \in [n-1]$, then w avoids the non-split pattern u in the usual sense.

We now go over some notation and properties of W as a Coxeter group. Let $S = \{s_1, \ldots, s_{n-1}\}$ denote the set of simple generators of W. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the length function and $\leq$ denote the Bruhat partial order on W. For any $w \in W$, define

$$S(w) := \{s \in S \mid s \leq w\}$$
$$D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$$
$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$$

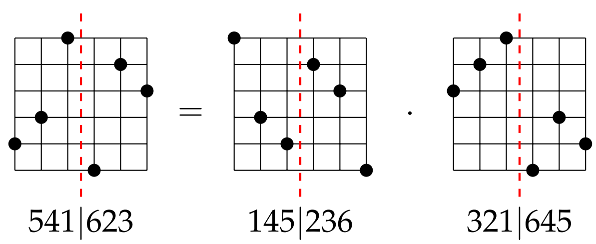
to be the **support, left descent set, and right descent set** of w, respectively. For any subset $J \subseteq S$, let W_J denote the parabolic subgroup generated by J and let W^J denote the set of minimal length coset representatives of W/W_J. For each $w \in W$ and $J \subseteq S$, there is a unique **parabolic decomposition** $w = vu$ where $v \in W^J$ and $u \in W_J$. The parabolic decompositions with respect to $J = S \setminus \{s_r\}$ can be described explicitly in terms of split patterns.

Lemma 2.4. *Let $w = w_1|w_2 = w(1)\cdots w(r)|w(r+1)\cdots w(n) \in W$ and $w = vu$ be the parabolic decomposition with respect to $J = S \setminus \{s_r\}$. Then*

1. *$v = v_1|v_2$ where v_1 and v_2 respectively consist of the entries of w_1 and w_2 arranged in increasing order.*
2. *$u = u_1|u_2$ where u_1 and u_2 are respectively the unique permutations on $[1,r]$ and $[r+1,n]$ with relative orders of w_1 and w_2.*

Proof. The lemma follows from the fact that $D_R(v) \subseteq \{s_r\}$ and that $s_r \notin S(u)$. □

Example 2.5. *Let $w = 541|623$. If $w = vu$ is the parabolic decomposition with respect to $J = S \setminus \{s_3\}$, then $v = 145|236$ and $u = 321|645$.*



In the case $J = S \setminus \{s_r\}$, each $v \in W^J$ corresponds to a unique Schubert variety in the Grassmannian $\mathrm{Gr}(r,n)$. In particular, define the Schubert variety

$$X_v^J := \{V \in \mathrm{Gr}(r,n) \mid \dim(V \cap E_j) \geq r_v[i,j]\}.$$

Geometrically, restricting π_r to X_w gives the projection $\pi_r : X_w \twoheadrightarrow X_v^J$ where the generic fiber is isomorphic to the Schubert variety X_u. We now give a combinatorial characterization for when π_r is a fiber bundle.

Proposition 2.6. ([5, Theorem 3.3, Proposition 4.2]) *Let $w \in \mathfrak{S}_n$ and $r < n$. Let $w = vu$ be the parabolic decomposition with respect to $J = S(w) \setminus \{s_r\}$. Then the following are equivalent.*

1. *$w = vu$ is a **BP decomposition** with respect to J.*
2. *$S(v) \cap J \subseteq D_L(u)$.*
3. *The projection $\pi_r : X_w \twoheadrightarrow X_v^J$ is a Zariski-locally trivial X_u-fiber bundle.*

The equivalencies in Proposition 2.6 are proved in [5] and for this paper, we will take either parts (2) or (3) of Proposition 2.6 as the definition of BP decomposition (note that this definition corresponds to a "Grassmannian BP decomposition" in [5]). The goal of Theorem 1.1 is to give a pattern avoidance criteria on the permutation w for any of these equivalent conditions.

Finally, we say w has a **complete BP decomposition** if we can write $w = v_k \cdots v_1$ where for every $i \in [k-1]$, we have $|S(v_i \cdots v_1)| = i$ and $v_i(v_{i-1} \cdots v_1)$ is a BP decomposition with respect to $S \setminus \{s_{r_i}\}$ where s_{r_i} is the unique simple generator in $S(v_i) \setminus S(v_{i-1} \cdots v_1)$.

Observe that the maps $\pi_r = \pi^{[n-1]}_{\{r\}}$ are not of the form $\pi^{\sigma_i}_{\sigma_{i-1}}$ used in Definition 1.2. The next proposition gives the connection between BP decompositions and complete parabolic bundle structures on Schubert varieties. The proposition follows directly from [5, Lemma 4.3] and the proof of [5, Corollary 3.7].

Proposition 2.7. ([5, Lemma 4.3, Corollary 3.7]) *The permutation w has a complete BP decomposition if and only if X_w has a complete parabolic bundle structure.*

3 Proof of Main theorems

In this section we prove Theorems 1.1 and 1.4. We begin with two important well-known lemmas on permutations and leave the proofs as exercises.

Lemma 3.1. *Let $v = v(1) \cdots v(n) \in W^J$ where $J = S \setminus \{s_r\}$. Then*

$$S(v) = \{s_k \in S \mid v(r+1) \leq k < v(r)\}.$$

Lemma 3.2. *Let $u = u(1) \cdots u(n) \in W$. Then $D_L(u) = \{s_k \in S \mid u^{-1}(k+1) < u^{-1}(k)\}$.*

In the proofs of Theorems 1.1 and 1.4, we will often refer to sub-matrices or rectangular regions of a permutation matrix. Let A be the permutation matrix of $w = w(1) \cdots w(n)$. We say a region R of A is **empty** if the interior of R contains no nodes of the form $(w(i), i)$. We say a region R is **decreasing** if for every pair $(w(i), i), (w(j), j)$ in R, we have $i < j$ implies $w(i) > w(j)$. Empty regions in a permutation matrix will be denoted by a shaded background and decreasing regions with be decorated (counter intuitively) with a northeast arrow. Finally, we say a pair of nodes $(w(i), i), (w(j), j)$ are **increasing** if $i < j$ and $w(i) < w(j)$.

Proof of Theorem 1.1. Fix $r < n$ and let $w = w(1) \cdots w(n) \in W$. Let $w = vu$ be the parabolic decomposition with respect to $J = S \setminus \{s_r\}$. By Proposition 2.6, it suffices to prove that w avoids the split patterns $3|12$ and $23|1$ with respect to position r if and only if $S(v) \cap J = S(v) \setminus \{s_r\} \subseteq D_L(u)$. Note that if $S(v) = \emptyset$, then the theorem immediately follows and hence we will assume that v is not the identity.

Let
$$m := \max\{w(k) \mid k \leq r\} \qquad \text{and} \qquad l := \min\{w(k) \mid k > r\}.$$
The nodes $(m, w^{-1}(m))$ and $(l, w^{-1}(l))$ partition the permutation matrix of w into regions labeled $A - H$ as in Figure 1. By definition of m and l, the regions D and E must be empty. Moreover, Lemma 2.4 part (1) and Lemma 3.1 imply that
$$S(v) = \{s_k \mid l \leq k < m\}. \tag{3.1}$$

Similarly, the permutation matrix of u partitions into regions $A' - H'$ as in Figure 1. Observe that since v is not the identity, we have $l \leq r$. By Lemma 2.4 part (2), the nodes in each region labeled $A - H$ maintain the same relative order of those in $A' - H'$ respectively. In particular, $(r, w^{-1}(m))$ and $(r+1, w^{-1}(l))$ are nodes in the permutation matrix of u. Furthermore, since regions D and E are empty, the sizes of regions A and H are the same as the size of regions A' and H'.

Now suppose w avoids the patterns $3|12$ and $23|1$ with respect to position r. Then regions B, G must be empty and regions C, F must be decreasing in the permutation matrix of w. Thus regions B', G' are empty and regions C', F' are decreasing in the permutation matrix of u (See Figure 2). Now Lemma 3.2 and (3.1) imply that $D_L(u)$ contains $S(v) \setminus \{s_r\}$ and hence $w = vu$ is a BP decomposition.

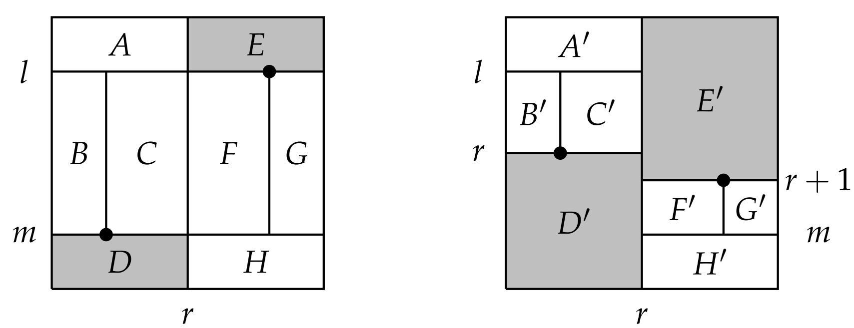


Figure 1: Permutation matrices of w and u partitioned by $(m, w^{-1}(m))$ and $(l, w^{-1}(l))$.

Conversely, suppose $S(v) \setminus \{s_r\} \subseteq D_L(u)$. In particular, Lemma 3.2 and (3.1) say that $u^{-1}(k+1) < u^{-1}(k)$ for all $k \in [l, r-1] \sqcup [r+1, m-1]$. This implies that regions B', G' are empty and regions C', F' are decreasing in the permutation matrix of u. Hence regions B, G are empty and regions C, F are decreasing in the permutation matrix of w. Thus w avoids both split patterns $3|12$ and $23|1$ with respect to position r. This completes the proof.

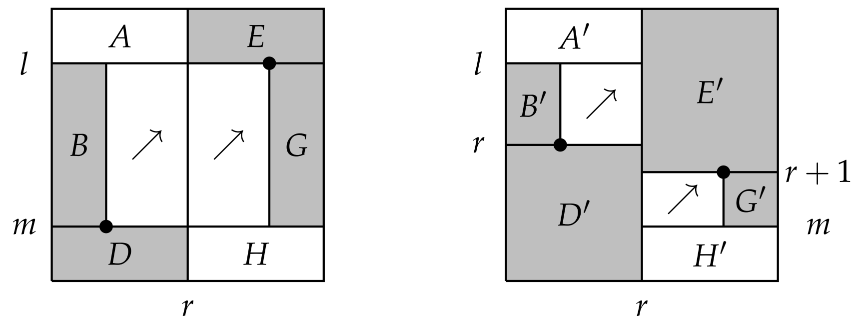


Figure 2: Permutation matrices of w and u with w avoiding $3|12$ and $23|1$ with respect to position r or equivalently, $S(v) \setminus \{s_r\} \subseteq D_L(u)$.

□

Proposition 3.3. *If $w \in W$ avoids 3412, 52341 and 635241, then there exists $r < n$ such that w avoids $3|12$ and $23|1$ with respect to position r. Furthermore, if $S(w) \neq \emptyset$, then we can choose r such that $s_r \in S(w)$.*

Proof. We prove the first part of Proposition 3.3 by contradiction. Suppose for every position $r < n$, w contains either $3|12$ or $23|1$. In particular, w contains $3|12$ with respect to position $r = 1$. Any $w(1)w(i)w(j)$ in relative position $3|12$ partitions the permutation matrix of w into regions labelled $A - K$ as in Figure 3. Moreover, we can choose nodes $(w(i), i), (w(j), j)$ such that regions E, F, J are empty. Since w avoids 3412, region D must also be empty and regions C and I must be decreasing.

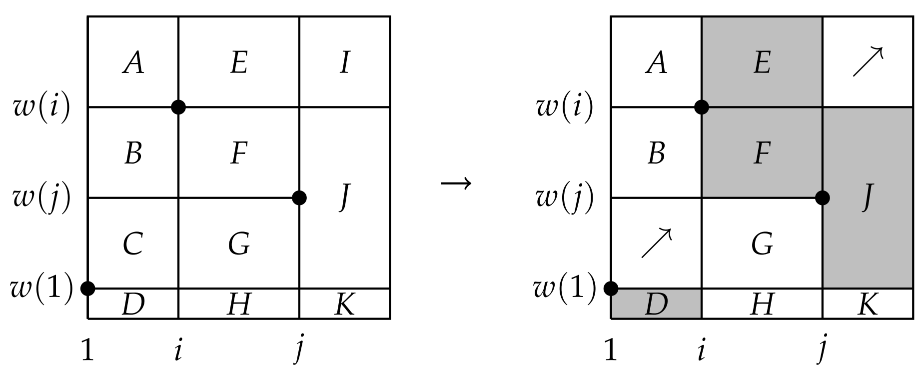


Figure 3: Permutation matrix of w containing $3|12$ with respect to position $r = 1$.

Now w contains either pattern $3|12$ or $23|1$ with respect to position $r = i$. We consider several cases depending on if region I is empty or nonempty and if w contains $3|12$ or $23|1$ with respect to position i.

Case 1: Suppose the region I is nonempty and w contains $3|12$ with respect to position i. Since regions D, E, F and J are empty and I is decreasing, the permutation matrix of

w must contain two increasing nodes in region G as in Figure 4. This implies w contains the pattern 52341 which is a contradiction.

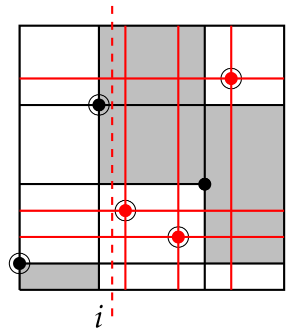


Figure 4: Permutation matrix of w containing $3|12$ with respect to $r = i$ and region I is nonempty.

Case 2: Suppose the region I is nonempty and w contains $23|1$ with respect to position i. If region A has a node belonging to the pattern $23|1$, then w contains the pattern 52341. Otherwise, since region C is decreasing, w must contain a pair of increasing nodes in region B or $B \cup C$. If the nodes are in region B, then w contains 52341 and if the nodes are in region $B \cup C$, then w contains 635241. See Figure 5 for an illustration of these three subcases.

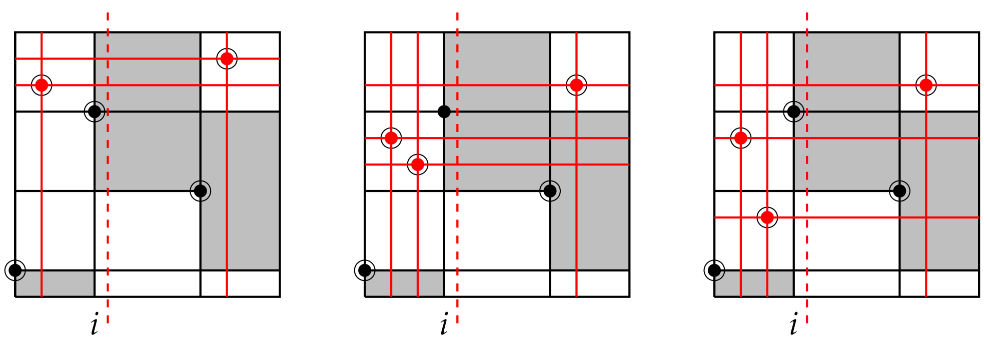


Figure 5: Permutation matrix of w containing $23|1$ with respect to $r = i$ and region I is nonempty.

Case 3: Suppose the region I is empty. Since region C is decreasing, it is not possible for w to contain $23|1$ with respect to position i. Hence w contains $3|12$ and thus region G must contain a pair of increasing nodes. These nodes partition region $G \cup H$ into subregions labeled $A' - K'$ as in Figure 6. Choose increasing nodes $(w(i'), i')$ and $(w(j'), j')$ in region G, so that regions E', F' and J' are empty. Also, since w avoids 3412 and 52341, we can further assume that regions A' and D' are empty and that regions C' and I' are decreasing.

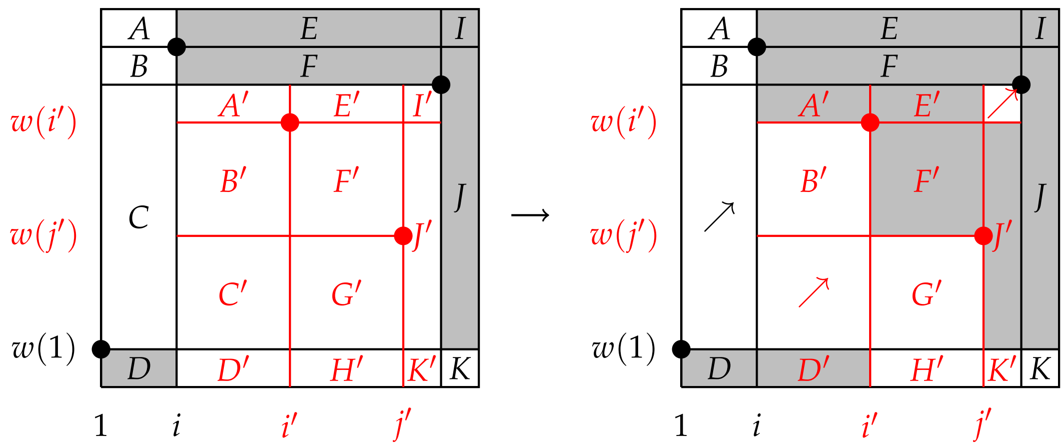


Figure 6: Permutation matrix of w containing $3|12$ with respect to position $r = i$ and region I is empty.

Now w contains $3|12$ or $23|1$ with respect to position $r = i'$. First, if w contains $3|12$, then, since region I' is decreasing, w must have a pair of increasing nodes in region G'. This implies w contains 52341.

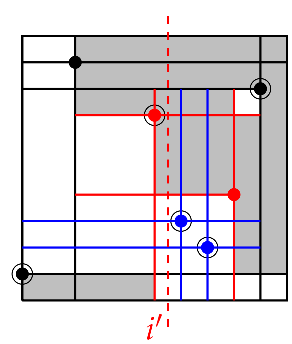


Figure 7: Permutation matrix of w containing $3|12$ with respect to position $r = i'$.

If w contains $23|1$, then the fact that regions C and C' are decreasing implies that w has a pair of increasing nodes in either regions $B', B' \cup C', C \cup B'$ or $C \cup C'$. If w contains increasing nodes in regions B' or $B' \cup C'$, then w contains 52341 or 635241 respectively as in Figure 8.

Finally, if w contains increasing nodes in regions $C \cup B'$ or $C \cup C'$, then we have the following three possibilities as in Figure 9.

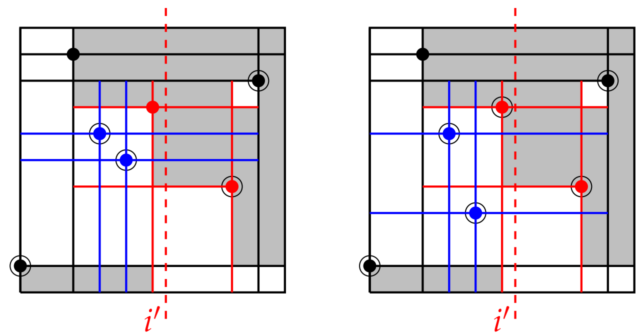


Figure 8: Permutation matrix of w containing $23|1$ with respect to position $r = i'$ using regions B' and $B' \cup C'$.

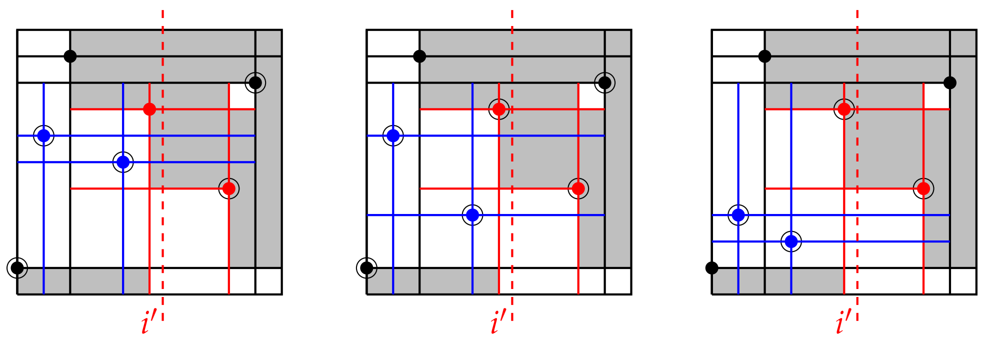


Figure 9: Permutation matrix of w containing $23|1$ with respect to position $r = i'$ using regions $C \cup B'$ and $C \cup C'$.

We can see that w contains 52341, 635241 and 3412 respectively for each of these possibilities. This completes the first part of the proof.

For the second part, if $w \in W$ avoids the patterns 3412, 52341 and 635241, then there exists $r < n$ where the parabolic decomposition $w = vu$ with respect to $J = S \setminus \{s_r\}$ is a BP decomposition. If $s_r \in S(w)$, then we are done. Otherwise, $s_r \notin S(w)$ which implies $w = u$. Write $w = w_1|w_2$ split at position r. If $J_1 = \{s_1, \cdots, s_{r-1}\}$ and $J_2 = J \setminus J_1$, then Lemma 2.4 implies that w_1 and w_2 also avoid 3412, 52341 and 635241 as permutations in $W_{J_1} \simeq S_r$ and $W_{J_2} \simeq S_{n-r}$ respectively. Since either r or $n - r$ is greater than 1 we will assume, without loss of generality, that $r > 1$ and $S(w_1) \neq \emptyset$. By induction, there exists $r' < r$ for which $s_{r'} \in S(w_1)$ and w_1 avoids $3|12$ and $23|1$ with respect to position r'. Now Lemma 2.4 implies w also avoids $3|12$ and $23|1$ with respect to position r'. But $S(w_1) \subseteq S(w)$ and hence $s_{r'} \in S(w)$. This completes the proof. □

Proof of Theorem 1.4. Theorem 1.4 follows directly by induction on the length of w using Proposition 3.3. □

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